CECS-PHY-04/02

Canonical sectors of five-dimensional Chern-Simons theories

Olivera Mišković[*,†], Ricardo Troncoso[*] and Jorge Zanelli[*]

[*]Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.
[†]Departamento de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile.

Abstract: The dynamics of five-dimensional Chern-Simons theories is analyzed. These theories are characterized by intricate self couplings which give rise to dynamical features not present in standard theories. As a consequence, Dirac's canonical formalism cannot be directly applied due to the presence of degeneracies of the symplectic form and irregularities of the constraints on some surfaces of phase space, obscuring the dynamical content of these theories. Here we identify conditions that define sectors where the canonical formalism can be applied for a class of non-Abelian Chern-Simons theories, including supergravity. A family of solutions satisfying the canonical requirements is explicitly found. The splitting between first and second class constraints is performed around these backgrounds, allowing the construction of the charge algebra, including its central extension.

1. Introduction

The best known gauge theories whose dynamical field is a connection on a fiber bundle are described by Yang-Mills and Chern-Simons (CS) actions. Three-dimensional CS theories are topological and also provide descriptions for gravitation and supergravity [1, 2]. There also exist gravity theories in higher odd dimensions described in terms of CS actions [3, 4]. For negative cosmological constant, in five dimensions the locally supersymmetric extension was found in [5], and for higher odd dimensions in [6, 7, 8]. For vanishing cosmological constant supergravity theories sharing this geometric structure have also been constructed in [9, 10, 11]. However, this elegant geometrical setting leads to a rich and quite complex dynamics. Indeed, for the purely gravitational sector, the Lagrangian in $D = 5$ dimensions, apart from the Einstein-Hilbert Lagrangian, also contains the Gauss-Bonnet term which is quadratic in the curvature, while for $D \geq 7$ additional terms with higher powers of the curvature and explicitly involving torsion are also required.

CS theories for $D \geq 5$ have been studied in different contexts (see *e. g.*, [12, 13, 14]), and are not necessarily topological theories but can contain propagating degrees of freedom [15]. Their dynamical structure depends on the location in phase space, and can drastically change from purely topological sectors to others with different numbers of local degrees of freedom. Sectors where the number of degrees of freedom is not maximal are *degenerate* and on them additional local symmetries emerge [16].

Furthermore, the symmetry generators in CS theories may be functionally dependent in some regions of phase space. Sectors where this happens are called *irregular* [17, 18]. Around irregular configurations the standard Dirac procedure, required to identify the physical observables (propagating degrees of freedom, conserved charges, etc.), is not directly applicable. Furthermore, the naive linearization of the theory fails to provide a good approximation to the full theory around irregular backgrounds [19, 20].

Degeneracy and irregularity are two independent conditions which may occur simultaneously in CS theories, and it is not yet fully understood how to deal with them. Irregular sectors are also found in the Plebanski theory [21]. Although these features are rarely found in field theories, they naturally arise in fluid dynamics, as in the description of vortices through the Burgers equation [22], or in transonic wave propagation in compressible fluids described by the Chaplygin and Tricomi equations [23].

The presence of degenerate and irregular sectors obscures the dynamical content of these theories, as Dirac's canonical formalism cannot be directly applied to them. In section 2, for simplicity we consider a non-Abelian CS theory in five dimensions, which captures the dynamical behavior without loss of generality. In section 3, we identify conditions that define canonical sectors, that is, those where the canonical formalism can be applied, and a family of solutions satisfying the canonical requirements is explicitly found. In Section 4, the splitting between first and second class constraints is performed around these backgrounds, allowing the construction of the charge algebra, including its central extension. Section 5 contains the discussion and outlook.

2. Dynamics

Chern-Simons Lagrangians describe gauge theories for a Lie-algebra-valued connection $\mathbf{A} = A_\mu^K \, \mathbf{G}_K \, dx^\mu$ where $K = 1, ..., \Delta$, and Δ is the dimension of the Lie group. The five-dimensional Chern-Simons form is such that its exterior derivative is an invariant 6-form,

$$dL = k \left\langle \mathbf{F}^3 \right\rangle = k \, g_{KLM} \, F^K \wedge F^L \wedge F^M , \tag{2.1}$$

where $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = F^K \mathbf{G}_K$ is the field strength 2-form and k is a dimensionless constant. Here $\langle \cdots \rangle$ stands for a symmetrized[1] trilinear invariant form, which defines the third rank invariant tensor $g_{KLM} = \langle \mathbf{G}_K \, , \mathbf{G}_L \, , \mathbf{G}_M \rangle$. The action is[2]

$$I\left[\mathbf{A}\right] = \int\limits_M L(\mathbf{A}) = k \int\limits_M \left\langle \mathbf{A}\mathbf{F}^2 - \frac{1}{2} \, \mathbf{A}^3 \mathbf{F} + \frac{1}{10} \, \mathbf{A}^5 \right\rangle , \tag{2.2}$$

where M is a five-dimensional manifold. The field equations read

$$\left\langle \mathbf{F}^2 \mathbf{G}_K \right\rangle = 0 \, . \tag{2.3}$$

2.1 Warming up with the linearized approximation

Non-Abelian CS theories are characterized by intricate self couplings which give rise to dynamical features not present in standard theories. These non-trivial properties can be captured in the linearized theory in five dimensions. The action in the linear approximation around a background solution $\bar{\mathbf{A}}$ has the form $I[\bar{\mathbf{A}} + \mathbf{u}] = I[\bar{\mathbf{A}}] + I_{\text{eff}}[\mathbf{u}]$, where the effective action is[3]

$$I_{\text{eff}}\left[\mathbf{u}\right] = 3k \int\limits_M \left\langle \mathbf{u} \bar{\mathbf{F}} \bar{\nabla} \mathbf{u} \right\rangle = \int d^5 x \, \left(\frac{1}{2} \, u_i^K \bar{\Omega}_{KL}^{ij} \bar{\nabla}_0 u_j^L - u_0^K C_K - h \right) , \tag{2.4}$$

with the "potential" $h\left(\mathbf{u}\right) \equiv 3k \, \varepsilon^{ijkl} g_{KLM} \, u_i^K \bar{F}_{0j}^L \bar{\nabla}_k u_l^M$ and the covariant derivative $\bar{\nabla} \mathbf{u} = d\mathbf{u} + \left[\bar{\mathbf{A}}, \mathbf{u} \right]$. The constraint C_K is given by

$$C_K = \bar{\Omega}_{KL}^{ij} \, \bar{\nabla}_i u_j^L , \tag{2.5}$$

and the kinetic term is defined by the symplectic form

$$\bar{\Omega}_{KL}^{ij} \equiv \Omega_{KL}^{ij}(\bar{F}) = -3k \, g_{KLM} \, \varepsilon^{ijkl} \bar{F}_{kl}^M , \tag{2.6}$$

which explicitly depends on the curvature.

Thus, the time evolution of the perturbations u_i^K depends on the background field strength, $\bar{F}_{kl}^K$, and hence the dynamics is crucially sensitive to the particular background

[1] In case of a superalgebra, the standard (anti)symmetrized form is assumed.

[2] Hereafter, wedge products between forms will be omitted.

[3] The five-dimensional manifold is assumed to be topologically $\mathbb{R} \otimes \Sigma$, and the coordinates are chosen as $x^\mu = (x^0, x^i)$, where x^i, with $i = 1, \ldots, 4$ correspond to the space-like section Σ.

around which it is explored. Overlooking this issue may lead to the paradoxical situation that the linearized theory around some backgrounds may seem to have more degrees of freedom than the fully nonlinear theory [19]. This is due to the fact that around those backgrounds, the linear approximation eliminates some constraints from the action.

Since the rank of $\bar{\Omega}$ is not fixed, the number of dynamical degrees of freedom can change throughout phase space:

Generic configurations have maximal rank of $\bar{\Omega}$ and form an open set in phase space. The theory around this kind of configurations has maximal number of degrees of freedom [15].

Degenerate configurations are the ones for which the rank of $\bar{\Omega}$ is not maximal so that they have additional gauge symmetries, and thus fewer degrees of freedom.

On the other hand, not only the rank of the symplectic form can vary, but the linear independence of the constraints C_K is not guaranteed either, and can fail on some backgrounds.

If the constraints C_K are independent, the sector is *regular.* This is the case in all standard theories. However, in CS theories, the constraints C_K can become dependent on some backgrounds, and these sectors are called *irregular.*

In an irregular configuration, there is always a linear combination of C's that identically vanishes. Consequently, in the linear approximation the number of degrees of freedom seems to increase in irregular sectors. However, this is an illusion induced by using the linear approximation which is no longer valid in this case. Moreover, the Dirac approach is not directly applicable in irregular sectors [24].

Indeed, careful analysis of the full non-linear theory shows that the number of degrees of freedom cannot increase. This is discussed in the next section.

2.2 Nonlinear dynamics

The field equations (2.3) can be written as

$$\varepsilon^{\mu\nu\lambda\rho\sigma} g_{KLM}\, F^{L}_{\mu\nu} F^{M}_{\lambda\rho} = 0\,. \tag{2.7}$$

Therefore, the field equations (2.7) split into the dynamical equations,

$$\Omega^{ij}_{KL}\, F^{L}_{0j} = 0\,, \tag{2.8}$$

and the constraints,

$$C_K = \frac{1}{4}\, \Omega^{ij}_{KL}\, F^{L}_{ij} \approx 0\,. \tag{2.9}$$

The symplectic matrix $\Omega^{ij}_{KL}(F)$, defined in Eq. (2.6), is a $4\Delta \times 4\Delta$ array with indices $\binom{i}{K}$ and $\binom{j}{L}$ with at least four zero modes (since $\Omega^{ij}_{KL} F^{L}_{jk} = \delta^{i}_{k} C_K \approx 0$), corresponding to the spatial diffeomorphisms. The existence of these four zero modes implies that the rank of Ω cannot exceed $(4\Delta - 4)$ [15].

As in the linearized approximation, the symplectic matrix is a function of the field strength F^{K}_{ij}, its rank is not necessarily constant throughout phase space. A configuration is

said to be *generic* if Ω has maximum rank, $4\Delta - 4$. Configurations of lower rank are *degenerate*; they have additional gauge symmetries and, consequently, fewer degrees of freedom. For example, any "vacuum" solution $F^K = 0$ is maximally degenerate since the symplectic form vanishes on it and hence no local excitations can propagate around these configurations. Since the rank cannot change under small deformations, generic sectors form open sets in field space, whereas degenerate configurations define sets of measure zero.

Regular configurations are those on which the constraints $C_K = 0$ are functionally independent, that is, they define a smooth surface with a unique tangent space on an open set around the configuration.[4] This means that the variations

$$\delta C_K = \frac{1}{2}\,\Omega^{ij}_{KL}\,\delta F^L_{ij}\,, \tag{2.10}$$

at a regular configuration, are Δ linearly independent vectors in phase space. Consequently, regularity is satisfied if the Jacobian of Eq. (2.10) has maximal rank, Δ, where now Ω^{ij}_{KL} has to be regarded as a $\Delta \times 6\Delta$ matrix with indices $\left(_K\right)$ and $\binom{ij}{L}$.

Note that genericity does not imply regularity, and vice-versa. This is because in one case the components Ω are regarded as the entries of a square matrix and, in the other, as the entries of a rectangular one.

Dirac's Hamiltonian formalism requires that, in an open set, the symplectic matrix be of constant rank and the constraints be functionally independent. Hence, we call *canonical sectors* of phase space those that are simultaneously generic and regular, because that is where the canonical formalism applies without modifications. Around degenerate or irregular configurations, the dynamical content of the theory requires extending Dirac's formalism as in [16, 18].

3. Canonical sectors

The identification of the canonical sectors for a Chern-Simons theory in general is a nontrivial task. However, a little bit of information about the group and the invariant tensor allows, in some cases, to identify these sectors. Let us split the generators as $\mathbf{G}_K = \{\mathbf{G}_{\bar{K}}, \mathbf{Z}\}$. If the group admits a third rank invariant tensor g_{KLM} such that $g_{\bar{K}\bar{L}z} := g_{\bar{K}\bar{L}}$ is invertible, and $g_{\bar{K}zz} = 0$, then the search for canonical sectors if much simpler. These conditions are trivially fulfilled by a non-Abelian group of the form $G = \tilde{G} \otimes U(1)$, and also apply to a larger class of theories including supergravity, for which the relevant group is super AdS_5, $SU(2,2|N)$.

Thus, the symplectic matrix reads

$$\Omega^{ij}_{KL} = \begin{pmatrix} \Omega^{ij}_{\bar{K}\bar{L}} & \Omega^{ij}_{\bar{K}z} \\ \Omega^{ij}_{\bar{L}z} & \Omega^{ij}_{zz} \end{pmatrix} = -3k\,\varepsilon^{ijkl} \begin{pmatrix} g_{\bar{K}\bar{L}\bar{M}}\,F^{\bar{M}}_{kl} + g_{\bar{K}\bar{L}}\,F^z_{kl} & g_{\bar{K}\bar{M}}\,F^{\bar{M}}_{kl} \\ g_{\bar{L}\bar{M}}\,F^{\bar{M}}_{kl} & \alpha F^z_{kl} \end{pmatrix}, \tag{3.1}$$

where $\alpha := g_{zzz}$.

[4] Regular configurations also form open sets in field space, while irregular ones form sets of measure zero.

Consider the following class of configurations,

$$\Omega^{ij}_{\bar{K}\bar{L}} = \text{non-degenerate}, \qquad \det(F^z_{ij}) \neq 0\,. \tag{3.2}$$

For this kind of configurations, the rank $\Re(\Omega_{KL}) = \Re(\Omega_{\bar{K}\bar{L}}) = 4\Delta - 4$, and therefore, they provide generic backgrounds.[5]

Among the configurations (3.2), the regular ones are those for which the variations of the constraints (2.10) are linearly independent, and this depends on the value of α,

$$\delta C_{\bar{K}} = \frac{1}{2}\,\Omega^{ij}_{\bar{K}\bar{L}}\,\delta F^{\bar{L}}_{ij} + \frac{1}{2}\,\Omega^{ij}_{\bar{K}z}\,\delta F^z_{ij}\,, \qquad \delta C_z = \frac{1}{2}\,\Omega^{ij}_{\bar{K}z}\,\delta F^{\bar{K}}_{ij} - \frac{3}{2}\,k\alpha\,\varepsilon^{ijkl} F^z_{ij}\,\delta F^z_{kl}\,. \tag{3.3}$$

If the $(\Delta-1)\times 6(\Delta-1)$ block $\Omega^{ij}_{\bar{K}\bar{L}}$ is non-degenerate, $\delta C_{\bar{K}}$ represent $\Delta - 1$ linearly independent vectors expressed as a linear combination of $\delta F^{\bar{K}}$. The problem of regularity then, reduces to the question of linear independence of the vector δC_z relative to the δC_K's.

If $\alpha \neq 0$, the block Ω_{zz} is non-vanishing, so that δC_z in (3.3) always contains the term proportional to δF^z, giving a vector linearly independent from $\delta C_{\bar{K}}$. Therefore, one concludes that

For $\alpha \neq 0$, the dynamics of the sectors defined by open sets around configurations of the form (3.2) *is always canonical.*

However, for $\alpha = 0$, the variations of C_z do not depend on F^z but on the remaining components $F^{\bar{K}}$. In particular, note that for a configuration with $F^{\bar{K}} = 0$, the block $\Omega^{ij}_{\bar{K}\bar{L}} = -3k\,\varepsilon^{ijkl} g_{\bar{K}\bar{L}}\,F^z_{kl}$ is invertible and hence, this configuration is generic. However, this configuration is irregular because $\delta C_z = 0$. One therefore concludes that,

For $\alpha = 0$, the theory contains additional irregular sectors which do not exist if $\alpha \neq 0$. Thus, a necessary condition to ensure regularity for configurations of the form (3.2)*, for $\alpha = 0$, is that at least one component of $F^{\bar{K}}$ does not vanish.*

In a canonical sector the counting of degrees of freedom can be safely done following the standard procedure [24], and in this case the number is $N = \Delta - 2$ [15].

A simple example of a solution of the constraints in the canonical sector is one whose only nonvanishing components of $F^{\bar{K}}$ is

$$F^{\bar{K}}_{12}\,dx^1 \wedge dx^2 \neq 0\,, \tag{3.4}$$

for at least one $\bar{K}$ and

$$F^z_{34} = 0\,, \qquad \text{with} \qquad \det\left(F^z_{ij}\right) \neq 0\,. \tag{3.5}$$

[5] This can be explicitly seen from $\Re\left(\Omega^{ij}_{KL}\right) = \Re\left(\Omega^{ij}_{\bar{K}\bar{L}}\right) + \Re\left(\Sigma^{ij}\right)$, where the matrix Σ^{ij} vanishes for $C_M \approx 0$, as it is given by $\Sigma^{ij} = \left(\delta^i_k C_z - \Omega^{il}_{z\bar{K}}\left(\Omega^{-1}\right)^{\bar{K}\bar{L}}_{lk} C_{\bar{L}}\right)\varepsilon^{jknm} F^z_{nm}$, and Ω^{-1} is the inverse of the invertible block $\Omega^{ij}_{\bar{K}\bar{L}}$ only.

4. Constraints and charge algebra

The canonical sectors satisfy all conditions necessary for the Dirac formalism to apply. That is, the first and second class constraints can be clearly defined and the counting and identifying of degrees of freedom can be explicitly done [25]. The explicit separation between first and second class constraints might be extremely difficult or impossible to carry out. This obstacle prevents, among other things, the canonical construction of the conserved charges.

The advantage of the class of canonical sectors described above, is that this splitting can be performed explicitly and, as a consequence, the conserved charges and their algebra are obtained following the Regge-Teitelboim approach [26].

4.1 Hamiltonian structure

The Hamiltonian formalism applied to CS systems was performed in [15] and can be easily extended to the supersymmetric case [19]. The action in Hamiltonian form reads

$$I\left[A\right]=\int d^{5}x\,\left(\mathcal{L}_{K}^{i}\,\dot{A}_{i}^{K}-A_{0}^{K}C_{K}\right)\,, \tag{4.1}$$

where the constraints C_K are defined in (2.10),

$$\mathcal{L}_{K}^{i}\equiv k\,\varepsilon^{ijkl}\,g_{KLM}\left(F_{jk}^{L}A_{l}^{M}-\frac{1}{4}\,f_{NS}{}^{M}A_{j}^{L}A_{k}^{N}A_{l}^{S}\right)\,, \tag{4.2}$$

and $f_{NS}{}^{M}$ are the structure constants of the Lie group. Additional constraints arise from the definition of the momenta,

$$\phi_{K}^{i}=\pi_{K}^{i}-\mathcal{L}_{K}^{i}\approx0\,, \tag{4.3}$$

and they satisfy

$$\left\{\phi_{K}^{i},\phi_{L}^{j}\right\}=\Omega_{KL}^{ij}\,. \tag{4.4}$$

Since the symplectic matrix Ω_{KL}^{ij} has at least four null eigenvectors, some ϕ's are first class and the explicit separation cannot be performed in general. However, for the class of canonical configurations described in the previous chapter, it is possible to separate them as

$$\begin{aligned}
&\text{First class}: && G_{K}=-C_{K}+\nabla_{i}\phi_{K}^{i}\approx0\,,\\
& && \mathcal{H}_{i}=F_{ij}^{z}\left(\phi_{z}^{j}-\phi_{\bar{K}}^{k}\left(\Omega^{-1}\right)_{kl}^{\bar{K}\bar{L}}\,\Omega_{\bar{L}z}^{lj}\right)=F_{ij}^{K}\phi_{K}^{j}\approx0\,,\\
&\text{Second class}: && \phi_{\bar{K}}^{i}\approx0\,,
\end{aligned} \tag{4.5}$$

where $\nabla_{i}\phi_{K}^{i}=\partial_{i}\phi_{K}^{i}+f_{KL}{}^{M}A_{i}^{L}\phi_{M}^{i}$, is the covariant derivative, and the constraints G_K satisfy the algebra

$$\left\{G_{K},G_{L}\right\}=f_{KL}{}^{M}\,G_{M}\,,\qquad\qquad\left\{G_{K},\phi_{L}^{i}\right\}=f_{KL}{}^{M}\,\phi_{M}^{i}\,. \tag{4.6}$$

The constraints G_K and $\mathcal{H}_i$ are generators of gauge transformations and improved spatial diffeomorphisms, respectively.[6] The second class constraints can be eliminated by introducing Dirac brackets,

$$\{X,Y\}^* \equiv \{X,Y\} - \int\limits_{\Sigma} d^4x \, \{X, \phi^i_{\bar{K}}(x)\} \left(\Omega^{-1}\right)^{\bar{K}\bar{L}}_{ij}(x) \left\{\phi^j_{\bar{L}}(x), Y\right\} , \tag{4.7}$$

and the reduced phase space is parametrized by $\left\{A^{\bar{K}}_i, A^z_i, \pi^i_z\right\}$.

4.2 Conserved charges

The separation (4.5) allows the construction of the conserved charges and the algebra following the Regge-Teitelboim approach [26]. The symmetry generators are

$$G_Q[\lambda] = G[\lambda] + Q[\lambda] \, , \tag{4.8}$$

where

$$G[\lambda] = \int\limits_{\Sigma} d^4x \, \lambda^K G_K \, , \tag{4.9}$$

and $Q[\lambda]$ is a boundary term such that $G_Q[\lambda]$ has well-defined functional derivatives. According to the Brown-Henneaux theorem, in general the charge algebra is a central extension of the gauge algebra [27],

$$\{Q[\lambda], Q[\eta]\}^* = Q[[\lambda,\eta]] + C[\lambda,\eta] \, , \tag{4.10}$$

where $[\lambda,\eta]^K = f_{LM}{}^K \lambda^L \eta^M$.

Thus, being in a canonical sector allows writing the charges as (see Appendix),[7]

$$Q[\lambda] = -3k \int\limits_{\partial\Sigma} g_{KLM} \, \lambda^K \bar{F}^L A^M, \tag{4.11}$$

where $\bar{F}$ is the background field strength and the parameters $\lambda^K(x)$ approach covariantly constant fields at the boundary, and the central charge is

$$C[\lambda,\eta] = 3k \int\limits_{\partial\Sigma} g_{KLM} \, \lambda^K \bar{F}^L d\eta^M. \tag{4.12}$$

The charge algebra can be recognized as the WZW$_4$ extension of the full gauge group [28]. In an irregular sector the charges are not well defined and the naive application of the Dirac formalism would at best lead to a charge algebra associated to a subgroup of G.

[6] An improved diffeomorphism, $\delta_\xi A^K_\mu = F^K_{\mu\nu}\,\xi^\nu$, differs from the Lie derivative by the gauge transformation with $\lambda^K = -\xi^\mu A^K_\mu$.

[7] Hereafter, the forms are defined at the spatial section Σ.

5. Discussion

Configurations in the canonical sectors satisfy all necessary conditions to safely apply the Dirac formalism to five-dimensional CS theories. The identification of these sectors in CS theories considered here, allows the explicit separation of first and second class constraints. Consequently, the conserved charges and their algebra are constructed following the Regge-Teitelboim approach.

As a direct application of these results in the context of supergravity, canonical BPS states saturating a Bogomol'nyi bound for the conserved charges (4.11) can be constructed [29]. One should expect that these results extend to higher odd dimensions. Indeed, conserved charges have been found in the purely gravitational case following a different approach [30].

Overlooking regularity obstructs the construction of well defined canonical generators. Consider theories with $\alpha = 0$ which accept simple generic configurations of the form (3.2), given by $F^{\bar{K}} = 0$ with $\det(F^z) \neq 0$. These configurations are generic but irregular (and therefore not canonical), since δC_z in Eq. (3.3) identically vanishes. If one naively chooses a configuration of this type as a background in the expression for the charges (4.11), one obtains that the $U(1)$ charge identically vanishes. This example simply reflects the fact that, for irregular configurations, $U(1)$ generator becomes functionally dependent, so that the naive application of Dirac's formalism within irregular sectors would lead to ill-defined expression for the charges.

Canonical sectors represent typical initial configurations around which one can prepare the system to let it evolve. In its evolution, the system may reach degenerate or irregular configurations. Although either degenerate or irregular configurations are easily identified, it is not yet fully understood how to deal with the dynamics around them in general, and it is clear that one must proceed with caution. However, the analysis of degenerate mechanical systems provide simple models that describe irreversible processes in which a system may evolve into a configuration with fewer degrees of freedom [16]. From these results one infers that, under certain conditions, a CS system may fall into a degenerate state from which it can never escape, losing degrees of freedom in an irreversible manner [16]; or it can also pass through irregular states unharmed [18].

The possibility that the fate of an initial configuration in a canonical sector of a higher dimensional CS system could be to end in a degenerate state, leads to an interesting effect: In Ref. [11], a new kind of eleven-dimensional supergravity was constructed as a CS system for the M-algebra. It was observed that the theory admits a class of vacuum solutions of the form $S^{10-d} \times Y_{d+1}$, where Y_{d+1} is a warped product of R with a d-dimensional spacetime. Remarkably, it was found that a nontrivial propagator for the graviton exists only for $d = 4$ and positive cosmological constant, and that perturbations of the metric around this solution reproduce linearized General Relativity around four-dimensional de Sitter spacetime.

Since this solution is a degenerate state one may regard it as the final stage of a wide class of initial canonical configurations that underwent a sort of dynamical dimensional reduction.

Acknowledgments

We gratefully acknowledge insightful discussions with M. Henneaux. This work was partially funded by FONDECYT grants 1020629, 1040921, 1051056, and 3040026. The generous support to CECS by Empresas CMPC is also acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

Appendix

On the reduced phase space $\phi^i_{\bar{K}} = 0$, the generators are

$$G_{\bar{K}} = -C_{\bar{K}}\,, \qquad\qquad G_z = -C_z + \partial_i \phi^i_z\,, \tag{5.1}$$

and the smeared generators become

$$G\left[\lambda\right] = 3k \int\limits_{\Sigma} \left\langle \lambda \mathbf{F}^2 \right\rangle + \int\limits_{\Sigma} \left\langle \lambda^z d\mathbf{\Phi} \right\rangle\,, \tag{5.2}$$

where the 3-form $\mathbf{\Phi} = \pi - k\,\left(\{\mathbf{A},\mathbf{F}\} - \frac{1}{2}\,\mathbf{A}^3\right)$ is dual of the constraints, *i. e.*, $\mathbf{\Phi}_{jkl} = \phi^i_z\,\varepsilon_{ijkl}$.

The variation of the generators (5.2) yields

$$\delta G\left[\lambda\right] = 6k \int\limits_{\Sigma} \left\langle \lambda \mathbf{F} \nabla \delta \mathbf{A} \right\rangle + \int\limits_{\Sigma} \left\langle \lambda^z d\delta\mathbf{\Phi} \right\rangle\,. \tag{5.3}$$

Integrating by parts this expression in order to obtain a well defined functional derivative, the leftover boundary term is the variation of the charge

$$\delta Q\left[\lambda\right] = -6k \int\limits_{\partial\Sigma} \left\langle \lambda \mathbf{F} \delta \mathbf{A} \right\rangle - 2k \int\limits_{\partial\Sigma} \left\langle d\lambda^z \mathbf{A} \delta \mathbf{A} \right\rangle - \int\limits_{\partial\Sigma} \left\langle \lambda^z \delta\mathbf{\Phi} \right\rangle\,. \tag{5.4}$$

This expression can be integrated out provided the connection is fixed at the boundary, as

$$\mathbf{A} \;\longrightarrow\; \bar{\mathbf{A}} \qquad \text{at } \partial\Sigma\,, \tag{5.5}$$

where $\bar{\mathbf{A}}$ is a background configuration in the canonical sector. Then, the charges are

$$Q\left[\lambda\right] = -6k \int\limits_{\partial\Sigma} \left\langle \lambda \bar{\mathbf{F}} \mathbf{A} \right\rangle - 2k \int\limits_{\partial\Sigma} \left\langle d\lambda^z \bar{\mathbf{A}} \mathbf{A} \right\rangle\,, \tag{5.6}$$

where the term proportional to $\mathbf{\Phi}$ vanishes on the constraint surface.

Note that the second term $\left\langle d\lambda^z \bar{\mathbf{A}} \mathbf{A} \right\rangle$ identically vanishes when it is evaluated on the background $\mathbf{A} = \bar{\mathbf{A}}$, since $g_{zKL}\,\bar{A}^K \bar{A}^L \equiv 0$.

Since the asymptotic behavior of the fields approaches a background configuration as $\mathbf{A} \sim \bar{\mathbf{A}} + \Delta\mathbf{A}$, the parameters of the asymptotic symmetries are of the form $\lambda \sim \bar{\lambda} + \Delta\lambda$. In particular, $\bar{\lambda}^z$ is a constant, and then

$$\left\langle d\lambda^z \bar{\mathbf{A}} \mathbf{A} \right\rangle \sim \left\langle d\left(\Delta\lambda^z\right) \bar{\mathbf{A}} \left(\Delta\mathbf{A}\right) \right\rangle\,, \tag{5.7}$$

is a subleading contribution which vanishes as it approaches the boundary. Therefore, the charges are given by (4.11)

$$Q\left[\lambda\right] = -6k \int\limits_{\partial\Sigma} \left\langle \lambda \bar{\mathbf{F}} \mathbf{A} \right\rangle . \tag{5.8}$$

References

[1] A. Achúcarro and P. K. Townsend, *Phys. Lett.* **B180** (1986) 89.

[2] E. Witten, *Nucl. Phys.* **B311** (1988) 46.

[3] A. Chamseddine, *Phys. Lett.* **B233** (1989) 291.

[4] R. Troncoso and J. Zanelli, *Class. Quant. Grav.* **17** (2000) 4451.

[5] A. Chamseddine, *Nucl. Phys.* **B346** (1990) 213.

[6] R. Troncoso and J. Zanelli, *Phys. Rev.* **D58**: R101703 (1998).

[7] R. Troncoso and J. Zanelli, *Int. J. Theor. Phys.* **38** (1999) 1181.

[8] R. Troncoso and J. Zanelli, "Chern-Simons Supergravities with Off-Shell Local Superalgebras", in *Black Holes and Structre of the Universe*, editors C. Teitelboim and J. Zanelli (World Scientific, Singapore, 1999). Archive: `hep-th/9902003`.

[9] M. Bañados, R. Troncoso and J. Zanelli, *Phys. Rev.* **D54** (1996) 2605.

[10] M. Hassaine, R. Olea and R. Troncoso, *Phys. Lett.* **B599** (2004) 111.

[11] M. Hassaine, R. Troncoso and J. Zanelli, *Phys. Lett.* **B596** (2004) 132; *Proc. Sci.* **WC2004** (2005) 006.

[12] R. Floreanini and R. Percacci, *Phys. Lett.* **B224** (1989) 291.

[13] R. Floreanini, R. Percacci and R. Rajaraman, *Phys. Lett.* **B231** (1989) 119.

[14] V. P. Nair and J. Schiff, *Phys. Lett.* **B246** (1990) 423; *Nucl. Phys.* **B371** (1992) 329.

[15] M. Bañados, L. J. Garay and M. Henneaux, *Phys. Rev.* **D53** (1996) 593; *Nucl. Phys.* **B476** (1996) 61.

[16] J. Saavedra , R. Troncoso and J. Zanelli, *J. Math. Phys.* **42** (2001) 4383; *Rev. Mex. Fis.* **48** (2002) SUPPL 387-89.

[17] M. Henneaux and C. Teitelboim, "Quantization of Gauge Systems", Univ. Pr. (1992) 520 p., Princeton, USA.

[18] O. Mišković and J. Zanelli, *J. Math. Phys.* **44** (2003) 3876; "Irregular Hamiltonian Systems", Proceedings of the XIII Chilean Symposium of Physics (Concepción, Chile, 2002). Archive: `hep-th/0301256`.

[19] O. Chandía, R. Troncoso and J. Zanelli, "Dynamical Content of Chern-Simons Supergravity", in *Trends in Theoretical Physics II*, H. Falomir, R .E. Gamboa Saravi and F. Schapopsnik, Eds. AIP Conf. Proceedings 484, 231-237, Amer. Inst. Phys., Woodbury, NY, 1999. Archive: `hep-th/9903204`.

[20] O. Mišković, "Dynamics of Wess-Zumino-Witten and Chern-Simons theories", Ph.D. Thesis (Universidad de Santiago de Chile, Jan. 2004). Archive: `hep-th/0401185`.

[21] E. Buffenoir, M. Henneaux, K. Noui and Ph. Roche, *Class. Quant. Grav.* **21** (2004) 5203.

[22] D. V. Choudnovsky and G. V. Choudnovsky, *Nuovo Cim.* **B40**, 339 (1977).

[23] Landau, L. D. and Lifschitz, *E. M. Fluid Mechanics*, 2nd ed. Oxford, England: Pergamon Press, 1982, Chapter XII.

[24] P. A. M. Dirac, Lectures on Quantum Mechanics, Yeshiva University, New York, 1964.

[25] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B **332**, 169 (1990).

[26] T. Regge and C. Teitelboim, *Ann. Phys. (N.Y.)* **88** (1974) 286.

[27] J. D. Brown and M. Henneaux, *J. Math. Phys.* **27** (1986) 489; *Commun. Math. Phys.* **104** (1986) 207.

[28] A. Losev, G. Moore, N. Nekrasov and S. Shatashvili, *Nucl. Phys. Proc. Suppl.* **46** (1996) 130.

[29] O. Mišković, R. Troncoso and J. Zanelli, "Dynamics and BPS states of AdS_5 Supergravity with Gauss-Bonnet term". Preprint CECS-PHY-05/07.

[30] P. Mora, R. Olea, R. Troncoso and J. Zanelli, JHEP **0406** (2004) 036.